Class 21 - §5.1b & §5.5 Applications of Exp / Log Equations

Review

→ Percentages & Decimals

Ex1: $8.5\% = 8.5 \div 100 = 0.085$

Ex2: the # $0.00043 = 0.0043\%$

→ Appoximate Exponential expressions with Calculator

Ex3: $1000\, e^{1.338} \approx 3811.4131$

→ Solve equations of the type $\boxed{b^u = c}$:

Ex4: $e^x = 5 \Rightarrow \cancel{\ln(e^x)} = \ln(5) \Rightarrow x = \ln 5 \approx 1.6094.$

Ex.5: $4^x = 21 \Rightarrow \cancel{\log_4(4^x)} = \log_4(21) \Rightarrow x = \log_4 21 \underset{\text{Change of base}}{=} \frac{\ln 21}{\ln 4} \approx 2.1962$

Ex6: $\frac{35\, e^{x+2}}{35} = \frac{10}{35} \Rightarrow e^{x+2} = \frac{2}{7} \Rightarrow \cancel{\ln(e^{x+2})} = \ln\left(\frac{2}{7}\right)$

$\Rightarrow \underset{-2}{x+2} = \underset{-2}{\ln\left(\frac{2}{7}\right)} \Rightarrow x = \ln\left(\frac{2}{7}\right) - 2 \Rightarrow x \approx -3.2528$

§ 5.1b Exponential functions

Periodic compound interest formula: $A = P\left(1+\frac{r}{n}\right)^{nt}$

A: amount, t: time (years), P: principal, r: annual rate (decimal)

Ex1: $P = 12{,}500$, $r = 9\%$, $n = 12$, $t = 10$. $A = ?$

$$A = 12500\cdot\left(1+\frac{0.09}{12}\right)^{12\cdot 10} \longrightarrow A = 30{,}641{,}96$$

Loan of P. Every month the debt increases by $0.75\% = \frac{9\%}{12}$.

After 1 month: $12500 \cdot 1.0075$

After 12 month: $12500\cdot(1.0075)^{12}$

After 10 years: $12500\cdot(1.0075)^{120}$

Ex2: Which investment is the greatest in total amount?

a) $\underbrace{\$3{,}000}_{\text{principal }(P)}$ invested for $\underbrace{6 \text{ years}}_{\text{time }(t)}$ compounded $\underbrace{\text{semiannually}}_{\#\text{ of periods }(n)}$ at $\underbrace{8\%}_{\text{rate }(r)}$.

$$A = P\left(1+\frac{r}{n}\right)^{nt} \Rightarrow A = 3000\left(1+\frac{0.08}{2}\right)^{2\cdot 6} \Rightarrow \boxed{A = 4{,}803.10}$$

3000 * 1.004 ^ 12

b) \$4,000 invested for 5 years compounded quarterly at 3.6%.

$$A = 4000\left(1+\frac{0.036}{4}\right)^{4\cdot 5} \Rightarrow A = 4{,}785.02$$

Continuous compounded interest formula: $\boxed{A = P\,e^{rt}}$

Ex1: $P = 12500$, $r = 9\%$, $t = 10 \Rightarrow A = 12500\,e^{0.09\cdot 10} \Rightarrow A = 30{,}745.04$

Ex 2: What is the amount of a \$9,000 investment with a rate of 5.75% compounded continuously after 3 years?

$A = 9000 \cdot e^{0.0575 \cdot 3} \Rightarrow A = 10,694.45.$

Growth Rate

$P(t) = P_0 e^{kt}$

Population after t (time)

P_0: initial population: $P(0) = P_0 e^{k \cdot 0} = P_0 e^0 = P_0 \cdot 1 = P_0$

k: constant of growth

Ex 1: The population of a rural city follows the growth model $P(t) = 3700 e^{0.029t}$, where t is the number of years after 1990.

a) What was the population in 1990?

$t = 0$. $P(0) = 3700 \cdot e^{0.029 \cdot 0} \Rightarrow P(0) = 3700 \cdot e^0 \Rightarrow P(0) = 3,700$.

b) What is the relative growth rate as a percentage?

$\underbrace{\text{relative growth rate}}_{\text{constant of growth } k}$ $k = 0.029$ or 2.9%

c) Use this model to approximate the population in 2032.

$t = 2032 - 1990 \Rightarrow t = 42$

$P(42) = 3700 \cdot e^{0.029 \cdot 42} \Rightarrow \boxed{P(42) = 12,508}$

Ex2: The relative growth of a certain bacteria colony is 25% per hour. Suppose there are 5 bacteria iniatially. Answer the following:

a) Find the function that describes the population of bacteria after "t" hours. $P(t) = P_0 e^{kt} \Rightarrow P(t) = 5e^{0.25t}$

b) How many bacteria should be expected after 1 day?

1 day = 24 hours. $P(24) = 5 \cdot e^{0.25 \cdot 24} \Rightarrow P(24) = 5 \cdot e^{6} = 2017$.

Exponential Decay

After 5,730 years the mass of Carbon-14 (^{14}C) will be half of the amount today.

Ex: An archeologist found a mummy with 20% of the original amount of C-14. Consider that the half-life of 5,730 years, answer the following:

a) What is the formula that describes the amount of C-14 after "t" years? $A(t) = A_0 e^{kt}$

b) What is the relative decay constant k?

A_0 is the initial amount of C-14.

half-life is 5730 years: $A(5730) = \frac{1}{2} A_0$ or $\frac{A_0}{2}$.

$A(5730) = A_0 e^{k \cdot 5730}$

$$\frac{\frac{\cancel{A_0}}{2}}{\cancel{A_0}} = \frac{\cancel{A_0} \cdot e^{5730k}}{\cancel{A_0}} \Rightarrow \frac{1}{2} = e^{5730k} \Rightarrow \ln\left(\frac{1}{2}\right) = \ln\left(e^{5730k}\right) \Rightarrow$$

$$\frac{\ln\left(\frac{1}{2}\right)}{5730} = \frac{5730k}{5730} \Rightarrow \boxed{k = \frac{\ln\left(\frac{1}{2}\right)}{5730}} \rightarrow \boxed{A(t) = A_0 e^{\frac{\ln\left(\frac{1}{2}\right)}{5730} \cdot t}}$$

c) How old is this mummy?

The mummy has $0.2A_0$ of C-14 today: $A(t) = 0.2A_0$

$$\frac{0.2A_0}{A_0} = \frac{A_0 e^{\frac{\ln\left(\frac{1}{2}\right)\cdot t}{5730}}}{A_0} \Rightarrow 0.2 = e^{\frac{\ln\left(\frac{1}{2}\right)}{5730}\cdot t} \Rightarrow$$

$$\ln(0.2) = \cancel{\ln}\left(\cancel{e}^{\frac{\ln\left(\frac{1}{2}\right)}{5730}\cdot t}\right) \Rightarrow \ln(0.2) = \frac{\ln\left(\frac{1}{2}\right)}{5730}\cdot t \Rightarrow$$

$$\frac{\ln(0.2)}{\frac{\ln\left(\frac{1}{2}\right)}{5730}} = \frac{\cancel{\frac{\ln\left(\frac{1}{2}\right)}{5730}}\cdot t}{\cancel{\frac{\ln\left(\frac{1}{2}\right)}{5730}}} \Rightarrow t = \frac{\ln(0.2)\cdot 5730}{\ln\left(\frac{1}{2}\right)} \Rightarrow \boxed{t = 13{,}305}$$

Ex2: A radioactive isotope has a half-life of 4 days. Estimate the percentage of the mass of this isotope after 3 days.

$$A(t) = A_0 e^{kt}.$$

1st: find k using the half-life. When $t=4$, $A(4) = \frac{1}{2}A_0$.

$$A(4) = A_0 e^{4k} \Rightarrow \frac{\frac{1}{2}\cancel{A_0}}{\cancel{A_0}} = \frac{\cancel{A_0} e^{4k}}{\cancel{A_0}} \Rightarrow \frac{1}{2} = e^{4k} \Rightarrow \ln\left(\frac{1}{2}\right) = \ln\left(e^{4k}\right)$$

$$\Rightarrow \frac{\ln\left(\frac{1}{2}\right)}{4} = \frac{4k}{4} \Rightarrow k = \frac{\ln\left(\frac{1}{2}\right)}{4} \Rightarrow k \approx -0.17 \rightarrow \boxed{A(t) = A_0 e^{\frac{\ln(1/2)}{4}\cdot t}}$$

2nd: compute $A(3)$. $A(3) = A_0 \cdot \underbrace{e^{\frac{\ln\left(\frac{1}{2}\right)}{4}\cdot 3}}_{0.5946} \Rightarrow A(3) = A_0 \cdot 0.5946$

After 3 days, the isotope still has 59.46% of its original amount.

Ex3: A radioactive isotope has leaked into a small stream. Five hundred years after the leak, 16% of the original amount of the substance remained. Determine the half-life of this radioactive isotope.

1^{st}: find k. $\quad A(t) = A_0 e^{kt}$

$\rightarrow$ $\underbrace{\text{After 500 years}}_{t=500}$, $\underbrace{\text{the remaining amount was 16\%}}_{A(500) = 0.16 A_0}$

$$A(500) = A_0 e^{500k} \Rightarrow \frac{0.16 \cancel{A_0}}{\cancel{A_0}} = \frac{\cancel{A_0} e^{500k}}{\cancel{A_0}} \Rightarrow 0.16 = e^{500k} \Rightarrow$$

$$\ln(0.16) = 500k \Rightarrow k = \frac{\ln(0.16)}{500}$$

2^{nd}: formula. $\quad A(t) = A_0 e^{\frac{\ln .16}{500} t}$

3^{rd}: half-life. $\quad A(t) = 0.5 A_0$ or $A(t) = \frac{A_0}{2}$

$$\frac{0.5 \cancel{A_0}}{\cancel{A_0}} = \cancel{\frac{A_0 e^{\frac{\ln(0.16)}{500} \cdot t}}{A_0}} \Rightarrow \ln(0.5) = \ln\left(e^{\frac{\ln(0.16)}{500} \cdot t}\right) \Rightarrow \ln(0.5) = \frac{\ln(0.16)\, t}{500}$$

$$t = \frac{500 \ln(0.5)}{\ln(0.16)} \Rightarrow t = 189 \text{ days.}$$